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Common fixed point theorems for cyclic contractive mappings in partial cone b -metric spaces and applications to integral equations*

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Abstract. In this paper, we introduce the concept of partial cone b -metric spaces as a generalization of partial metric, cone metric and b -metric spaces and establish some topological properties of partial cone b -metric spaces. Moreover, we also prove some common fixed point theorems for cyclic contractive mappings in such spaces. Our results generalize and extend the main results of Huang and Zhang [Cone metric spaces and fixed point theorems of contractive mappings, *J. Math. Anal. Appl.*, 332:1468–1476, 2007], Stanić et al. [Common fixed point under contractive condition of Ćirić's type on cone metric type spaces, *Fixed Point Theory Appl.*, 2012:35, 2012] and Latif et al. [Fixed point results for generalized (α, ψ) -Meir-Keeler contractive mappings and applications, *J. Inequal. Appl.*, 2014:68, 2014]. Some examples and an application are given to support the usability of the obtained results.

Keywords: partial metric spaces, cone metric spaces, b -metric spaces, contractive mappings, common fixed point.

1 Introduction and preliminaries

There are many generalizations of concept of metric spaces in the literature. In particular, many fixed point theorems transpose from metric spaces to b -metric spaces, partial metric spaces, cone metric spaces considered in the current literature.

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The concept of b -metric spaces was introduced by Bakhtin [7] and extensively used by Czerwik [8]. Since then, some interesting results have been presented about the existence of a fixed point for single-valued and multi-valued mappings in b -metric spaces (see [5, 6, 9, 15, 27]). In [21], Matthews introduced the concept of a partial metric space as a part of the study of denotational semantics of dataflow for networks. Moreover, Matthews showed that the Banach contraction principle could be generalized to the partial metric context for applications in program verification. After that, many fixed point results for mappings satisfying different contractive conditions in partial metric spaces have been proved (see [2, 4, 23]). Moreover, Shukla [30] defined the notion of partial b -metric spaces as a generalization of partial metric and b -metric spaces. Mustafa et al. [22] introduced a modified version of partial b -metric spaces in order to guarantee that each partial b -metric p_b generalizes a b -metric d_{p_b} .

In [11], Huang and Zhang introduced the notion of cone metric spaces and extended the Banach contraction principle to cone metric spaces over a normal solid cone. Moreover, they defined the convergence via interior points of the cone. Such an approach allows the investigation of the case that the cone is not necessarily normal. Since then, there were many references concerned with fixed point results in cone spaces (see [3, 13, 16, 20, 25, 26, 29, 32–34]). In 2011, Malhotra et al. [19] and Sonmez [31] defined a partial cone metric space; Hussain and Shah [12] introduced a cone b -metric space and established some topological properties in such spaces.

We first recall some definitions from b -metric spaces and partial metric spaces.

Definition 1. (See [7].) Let X be a nonempty set and $s \geq 1$ be a given real number. A function $d : X \times X \rightarrow \mathbb{R}^+$ is a b -metric on X if, for all $x, y, z \in X$, the following conditions hold:

- (b_1) $d(x, y) = 0$ if and only if $x = y$;
- (b_2) $d(x, y) = d(y, x)$;
- (b_3) $d(x, y) \leq s[d(x, z) + d(z, y)]$.

In this case, the pair (X, d) is called a b -metric space.

Definition 2. (See [21].) A partial metric on a nonempty set X is a function $p : X \times X \rightarrow \mathbb{R}^+$ such that for all $x, y, z \in X$:

- (p_1) $p(x, x) = p(x, y) = p(y, y)$ if and only if $x = y$;
- (p_2) $p(x, x) \leq p(x, y)$;
- (p_3) $p(x, y) = p(y, x)$;
- (p_4) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

In this case, the pair (X, p) is called a partial metric space and p is a partial metric on X .

Now, we recall some definitions from cone metric spaces.

Let E be a topological vector space. A cone of E is a nonempty closed subset P of E such that

- (i) $ax + by \in P$ for each $x, y \in P$ and each $a, b \geq 0$, and
- (ii) $P \cap (-P) = \{\theta\}$, where θ is the zero element of E .

Each cone P of E determines a partial order \preceq on E by $x \preceq y$ if and only if $y - x \in P$ for each $x, y \in E$. We shall write $x \prec y$ to mean $x \preceq y$ and $x \neq y$.

A cone P of a topological vector space E is solid if $\text{int } P \neq \emptyset$, where $\text{int } P$ is the interior of P . For each $x, y \in E$ with $y - x \in \text{int } P$, we write $x \ll y$. A cone P of a normed vector space $(E, \|\cdot\|)$ is normal if there exists $K > 0$ such that $\theta \preceq x \preceq y$ implies that $\|x\| \leq K\|y\|$ for each $x, y \in P$, and the minimal K is called a normal constant of P .

Definition 3. (See [11].) Let X be a nonempty set, and let P be a cone of a topological vector space E . A cone metric on X is a mapping $d : X \times X \rightarrow P$ such that for all $x, y, z \in X$:

- (d₁) $d(x, y) = \theta$ if and only if $x = y$;
- (d₂) $d(x, y) = d(y, x)$;
- (d₃) $d(x, y) \preceq d(x, z) + d(z, y)$.

The pair (X, d) is called a cone metric space over P .

It is obvious that b -metric spaces, partial metric spaces and cone metric spaces generalize metric spaces.

In [38], Zhu et al. introduced the following definition and extended the notion of a cyclic mapping in [17].

Definition 4. (See [38].) Let X be a nonempty set, m be a positive integer, A_1, A_2, \dots, A_m be subsets of X , $Y = \bigcup_{i=1}^m A_i$, and $S, T : Y \rightarrow Y$ be two self-maps. Then Y is said to be a cyclic representation of Y with respect to S and T if the following two conditions are satisfied:

- (i) $S(A_i), i = 1, 2, \dots, m$, are nonempty closed sets;
- (ii) $T(A_1) \subseteq S(A_2), T(A_2) \subseteq S(A_3), \dots, T(A_m) \subseteq S(A_1)$.

Let X be a nonempty set, $S, T : X \rightarrow X$ be two mappings. A point $x \in X$ is said to be a coincidence point of S and T if $Sx = Tx$. A point $w \in X$ is said to be a point of coincidence of S and T if, for some $x \in X$, $w = Sx = Tx$. The mappings S, T are said to be weakly compatible if they commute at their coincidence points (that is, $TSz = STz$, whenever $Sz = Tx$).

Lemma 1. (See [18].) Let E be a topological vector space and P be a cone, and $\{u_n\}$ be a sequence in E . Then $u_n \rightarrow_{\|\cdot\|} \theta$ implies that for each $c \in \text{int } P$, there exists a positive integer n_0 such that $c \pm u_n \in \text{int } P$, that is, $u_n \ll c$ for all $n \geq n_0$.

Recently, without using the normality of the cone, Malhotra et al. [19] and Jiang and Li [14] extended the results of [12] to θ -complete partial cone metric spaces. Latif et al. [18] presented a fixed point theorem for generalized (α, ψ) -Meir-Keeler contractive mappings in complete metric spaces. Meantime, other authors also obtained some interesting results in this area (see [1, 10, 17, 24, 28, 35–40]).

The aim of the paper is to introduce the concept of partial cone b -metric spaces and establish some topological properties of the partial cone b -metric spaces. Moreover, we

obtain some common fixed point results for cyclic α_S -Hardy–Rogers contractive mappings and generalized cyclic $(\alpha, \psi)_S$ -Meir–Keeler contractive mappings in such spaces. Also, we use one of our obtained results to prove an existence theorem of a common solution of integral equations. It is worth pointing that our results generalize and extend the main results of [1–4, 13, 16, 18, 26, 29, 32–34].

2 Basis definitions and properties of partial cone b -metric spaces

In this section, inspired by the notion of partial b -metric spaces in [22], we introduce the concept of a partial cone b -space and deduce some properties of partial cone b -metric spaces.

Definition 5. Let X be a nonempty set and P be a cone of a topological vector space E , and $s \geq 1$ be a given real number. A partial cone b -metric on X is a mapping $p : X \times X \rightarrow P$ such that for all $x, y, z \in X$:

- (p_1) $p(x, x) = p(x, y) = p(y, y)$ if and only if $x = y$;
- (p_2) $p(x, x) \preceq p(x, y)$;
- (p_3) $p(x, y) = p(y, x)$;
- (p_4) $p(x, y) \preceq s[p(x, z) + p(z, y) - p(z, z)] + (1 - s)[p(x, x) + p(y, y)]/2$.

The pair (X, p) is called a partial cone b -metric space with coefficient $s \geq 1$, and p is called a partial cone b -metric on X . Since $s \geq 1$, from (p_4) we have

$$p(x, y) \preceq s[p(x, z) + p(z, y) - p(z, z)] \preceq s[p(x, z) + p(z, y)] - p(z, z).$$

It should be noted that in a partial cone b -metric space (X, p) , if $p(x, y) = \theta$, then from (p_1) and (p_2) imply that $x = y$. But if $x = y$, $p(x, y)$ may not be θ . On the other hand, it is obvious that partial cone b -metric spaces generalize partial metric, cone metric and b -metric spaces. Now, we give some examples to illustrate the partial cone b -metric spaces.

Example 1. Let $X = E = C([0, T])$ and $P = \{u \in E : u(t) \geq 0 \text{ for all } t \in [0, T]\}$. Define a mapping $p : X \times X \rightarrow P$ by

$$p(x, y)(t) = f(t) \max_{0 \leq t \leq T} \{|x(t) - y(t)|^q + \beta\}$$

for all $x, y \in X$, where $q > 1$, $\beta \geq 0$, and $f : [0, T] \rightarrow \mathbb{R}^+$ is a function such that $f(t) = e^t$ for all $t \in [0, T]$. Then (X, p) is a partial cone b -metric space with coefficient $s = 2^{q-1}$.

Example 2. Let $E = C_R^1[0, 1]$ with the norm $\|u\| = \|u\|_\infty + \|u'\|_\infty$ and $X = P = \{u \in E : u(t) \geq 0, t \in [0, 1]\}$, which is a non-normal solid cone. Define a mapping $p : X \times X \rightarrow P$ by

$$p(x, y) = \begin{cases} x^2, & x = y, \\ (x + y)^2 & \text{otherwise.} \end{cases}$$

Then (X, p) is a partial cone b -metric space with coefficient $s = 3$.

The following conclusion is valid and its proofs is referenced to [33].

Proposition 1. Let (X, p) be a partial cone b -metric space with coefficient $s \geq 1$. For each $x \in X$ and each $c \gg \theta$, $c \in E$, let $B_p(x, c) = \{y \in X: p(x, y) \ll p(x, x) + c\}$ and $\mathbb{B} = \{B_p(x, c): x \in X, c \gg \theta\}$. Then \mathbb{B} is a subbase for some topology τ on X .

Proof. For each $c \gg \theta$, we have $p(x, x) \ll p(x, x) + c$ for all $x \in X$. It follows that $X = \bigcup_{B_p \in \mathbb{B}} B_p$. Hence, \mathbb{B} is a subbase for some topology τ on X . \square

We present the following example to show that \mathbb{B} is not a base for any topology on X .

Example 3. Let $X = \{i, j, k\}$ and put $p: X \times X \rightarrow \mathbb{R}^+$ as follows:

- (i) $p(i, i) = p(j, j) = 2$ and $p(k, k) = 1$;
- (ii) $p(i, j) = p(j, i) = 3$, $p(j, k) = p(k, j) = 2$ and $p(i, k) = p(k, i) = 6$.

It is clear that (X, p) is a partial cone b -metric space with coefficient $s = 3$. For each $x \in X$ and each $c > 0$, let $B_p(x, c) = \{y \in X: p(x, y) < p(x, x) + c\}$ and $\mathbb{B} = \{B_p(x, c): x \in X, c > 0\}$.

Since $p(i, j) = 3 < 2 + 1 = p(i, i) + 1$, we have $j \in B_p(i, 1)$. For any $c > 0$, since $p(j, k) = 2 < 2 + c = p(j, j) + c$, we have $k \in B_p(j, 1)$. On the other hand, $p(i, k) = 6 \geq 3 = 2 + 1 = p(i, i) + 1$ implies $k \notin B_p(i, 1)$. Hence, \mathbb{B} is not a base for any topology on X .

Let (X, p) be a partial cone b -metric space. In this paper, τ denotes the topology on X , \mathbb{B} denotes a subbase for the topology τ , and $B_p(x, c)$ denotes the p -ball in (X, p) , which are described in Proposition 1.

Definition 6. Let (X, p) be a partial cone b -metric space over a solid cone P of a normed vector space $(E, \|\cdot\|)$, $\{x_n\}$ be a sequence in X and $x \in X$.

- (i) $\{x_n\}$ converges to x with respect to p if, for every $c \in E$ with $c \gg \theta$, there exists $n_0 \in \mathbb{Z}^+$ such that for all $n > n_0$, $p(x_n, x) \ll p(x, x) + c$. We denote this by $x_n \rightarrow x$.
- (ii) $\{x_n\}$ is said to be a Cauchy sequence in (X, p) if there exists $u \in P$ such that $\lim_{n \rightarrow \infty} p(x_m, x_n) = u$. The partial cone b -metric space (X, p) is complete if each Cauchy sequence $\{x_n\}$ of X converges to a point $x \in X$ with respect to p such that $p(x, x) = u$.
- (iii) $\{x_n\}$ is said to be a θ -Cauchy sequence in (X, p) if, for each $c \in \text{int } P$, there exists a positive integer n_0 such that $p(x_n, x_m) \ll c$ for all $m, n \geq n_0$. The partial cone metric space (X, p) is θ -complete if each θ -Cauchy sequence $\{x_n\}$ of X converges to a point $x \in X$ with respect to p such that $p(x, x) = \theta$.

Note that if P is a normal solid cone of a normed vector space $(E, \|\cdot\|)$, then every θ -Cauchy sequence in (X, p) is Cauchy sequence (X, p) and each complete partial cone b -metric space is θ -complete.

On the other hand, in partial cone b -metric spaces, the limit of convergent sequence may not be unique. In fact, if (X, p) is a partial cone b -metric space, then (X, τ) is T_0 space, but need not be T_1 . Now, we give the following example.

Example 4. Let $E = \mathbb{R}^2$, $P = \{(x, y) \in E: x, y \geq 0\} \subset \mathbb{R}^2$ and $X = \mathbb{R}^+$. Define $p: X \times X \rightarrow P$ by

$$p(x, y) = \left(\frac{1}{2} \max\{x, y\}, \max\{x, y\} + \beta \right)$$

for all $x, y \in X$, where $\beta \geq 0$ is constant. Then (X, p) is a θ -complete partial cone b -metric space with coefficient $s = 1$. Now, we define a sequence $\{x_n\}$ in X by $x_n = 2$ for all $n \in \mathbb{N}$. Note that if $y \geq 2$, we have $p(x_n, y) = (y/2, y + \beta) = p(y, y)$. Thus, for every $c \gg \theta$, we have $\theta \preceq p(x_n, y) - p(y, y) \ll c$ for each $y \geq 2$ and for all $n \in \mathbb{N}$. Hence, $\{x_n\}$ converges to y with respect to p for all $y \geq 2$. Then the limit of convergent sequence in the partial cone b -metric space may not be unique. Now, we show that (X, p) is a T_0 space, but it is not a T_1 space. In fact, for any given $x_1, x_2 \in X$, $x_1 \neq x_2$. Suppose that $x_1 < x_2$ for any $c \gg \theta$, we have $B_p(x_1, c) = \{y \in X: p(x_1, y) \leq p(x_1, x_1) + c\} \supset [0, x_1]$ and $B_p(x_2, c) = \{y \in X: p(x_2, y) \leq p(x_2, x_2) + c\} \supset [0, x_2]$. Hence, $x_1 \in [0, x_1] \subset [0, x_2] \subset B_p(x_2, c)$, that is, (X, p) is not a T_1 space. Let $t = x_2 - x_1 > 0$, there exists $c_0 = (t/2, t/2)$ such that $x_2 \in B_p(x_1, c_0)$. Then (X, p) is a T_0 space.

3 Common fixed point theorems in partial cone b -metric spaces

In this section, we first give some properties of partial cone b -metric spaces. The following properties will be used (particularly when we deal with partial cone b -metric spaces in which the cone need not be normal).

Remark 1. Let (X, p) be a partial cone b -metric space over a solid cone P of a normed vector space $(E, \|\cdot\|)$. Then the following properties are used:

- (i) if $a \preceq b$ and $b \ll c$, then $a \ll c$;
- (ii) if $a \ll b$ and $b \preceq c$, then $a \ll c$;
- (iii) if $\theta \preceq u \ll c$ for each $c \in \text{int } P$, then $u = \theta$;
- (iv) if $c \in \text{int } P$, $a_n \rightarrow \theta$, then there exists a $k \in \mathbb{N}$ such that for all $n > k$, we have $a_n \ll c$;
- (v) if $a_n \preceq b_n$ and $a_n \rightarrow a$, $b_n \rightarrow b$, then $a \preceq b$ for each cone P ;
- (vi) if $a \preceq \lambda a$, where $0 \leq \lambda < 1$, then $a = \theta$.

Now, we introduce the concepts of generalized α -admissible mappings and cyclic α_S -Hardy–Rogers contractive mappings.

Definition 7. Let X be a nonempty set, $S, T: X \rightarrow X$ be two mappings and $\alpha: X \times X \rightarrow [0, +\infty)$ be a function. S and T are called generalized α -admissible if, for $x, y \in X$ such that $\alpha(Sx, Sy) \geq 1$, we have $\alpha(Tx, Ty) \geq 1$.

Example 5. Let $X = [0, \infty)$, $Sx = \log_2(1 + x)$ for all $x \in X$,

$$Tx = \begin{cases} \sqrt{x}, & x \in [0, 9], \\ x^2 + 2x, & x \in (9, \infty), \end{cases} \quad \text{and} \quad \alpha(x, y) = \begin{cases} 2, & x, y \in [0, 3], \\ 1/2 & \text{otherwise.} \end{cases}$$

In fact, if $x, y \in X$, $\alpha(Sx, Sy) = \alpha(\log_2(1+x), \log_2(1+y)) \geq 1$, then $x, y \in [0, 7] \subseteq [0, 9]$. Hence, for $x, y \in [0, 7]$, we have $Tx, Ty \in [0, \sqrt{7}] \subseteq [0, 3]$, and so $\alpha(Tx, Ty) \geq 1$. Thus, S and T are generalized α -admissible.

Definition 8. Let (X, p) be a partial cone b -metric space with coefficient $s \geq 1$, $\alpha : X \times X \rightarrow [0, +\infty)$ be a symmetric function. Let m be a positive integer, A_1, A_2, \dots, A_m be nonempty subsets of X , $Y = \bigcup_{i=1}^m A_i$, and $S, T : Y \rightarrow Y$ be two mappings. Then $T : Y \rightarrow Y$ is said to be cyclic α_S -Hardy-Rogers contractive if Y is a cyclic representation of Y with respect to S and T for any x and y lying in different adjacently labeled sets A_i and A_{i+1} , $i = 1, 2, \dots, m$,

$$\begin{aligned} \alpha(Sx, Sy)p(Tx, Ty) &\preccurlyeq a_1p(Sx, Sy) + a_2p(Tx, Sx) + a_3p(Ty, Sy) \\ &\quad + \frac{a_4}{s}p(Tx, Sy) + \frac{a_5}{s}p(Ty, Sx), \end{aligned} \quad (1)$$

where $A_{m+1} = A_1$ and $a_i \geq 0$ for $i = 1, 2, \dots, 5$ such that $2a_1s + (s+1)(a_2 + a_3 + a_4 + a_5) < 2$.

Theorem 1. Let (X, p) be a θ -complete partial cone b -metric space over a solid cone P of a normed vector space $(E, \|\cdot\|)$, m be a positive integer, A_1, A_2, \dots, A_m be nonempty subsets of X , $Y = \bigcup_{i=1}^m A_i$, $T : Y \rightarrow Y$ be a cyclic α_S -Hardy-Rogers contractive mapping satisfying (1). Suppose that the following conditions hold:

- (i) S and T are generalized α -admissible;
- (ii) there exists $x_0 \in A_1$ such that $\alpha(Sx_0, Tx_0) \geq 1$;
- (iii) if $\{y_n\}$ is a sequence in X such that $\alpha(y_n, y_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $y_n \rightarrow y$ as $n \rightarrow \infty$, then $\alpha(y_n, y) \geq 1$ for sufficiently large n .

Then S and T have a coincidence point in X , that is, there exists $z \in X$ such that $Sz = Tz$. Moreover, if S, T are weakly compatible and Λ is the set of coincidence points of S and T for all $x, y \in \Lambda$, we have $\alpha(Sx, Sy) \geq 1$, then S and T has a unique common fixed point in X .

Proof. Since $T(A_1) \subset S(A_2)$ and $x_0 \in A_1$, there exists an $x_1 \in A_2$ such that $Sx_1 = Tx_0$. Since $T(A_2) \subset S(A_3)$ and $x_1 \in A_2$, there exists an $x_2 \in A_3$ such that $Sx_2 = Tx_1$. Continuing this process, we can construct two sequences $\{x_n\}$ and $\{y_n\}$ defined by $y_{n+1} = Sx_{n+1} = Tx_n$ for all $n \in \mathbb{N}$, and there exists $i_n \in \{1, 2, \dots, m\}$ such that $x_n \in A_{i_n}$ and $x_{n+1} \in A_{i_{n+1}}$.

By condition (ii) we get $\alpha(Sx_0, Sx_1) = \alpha(Sx_0, Tx_0) \geq 1$. It follows from (i) that $\alpha(Tx_0, Tx_1) = \alpha(Sx_1, Sx_2) \geq 1$. By induction we have

$$\alpha(y_n, y_{n+1}) = \alpha(Sx_n, Sx_{n+1}) \geq 1 \quad \text{for all } n \in \mathbb{N}. \quad (2)$$

Since α is symmetric, we have $\alpha(y_{n+1}, y_n) = \alpha(Sx_{n+1}, Sx_n) \geq 1$ for all $n \in \mathbb{N}$.

Without loss of generality, assume that $y_{n+1} \neq y_n$ for all $n \in \mathbb{N}$ (otherwise, $Tx_{n_0} = Sx_{n_0+1} = y_{n_0+1} = y_{n_0} = Sx_{n_0}$ for some $n_0 \in \mathbb{N}$, then x_{n_0} is the coincidence point of S and T . Hence, the conclusion holds).

Since $x_n \in A_{i_n}$ and $x_{n+1} \in A_{i_{n+1}}$, by (1) and (2) and using the triangular inequality, we have

$$\begin{aligned}
 p(y_{n+1}, y_{n+2}) &= p(Tx_n, Tx_{n+1}) \preceq \alpha(Sx_n, Sx_{n+1})p(Tx_n, Tx_{n+1}) \\
 &\preceq a_1p(Sx_n, Sx_{n+1}) + a_2p(Tx_n, Sx_n) + a_3p(Tx_{n+1}, Sx_{n+1}) \\
 &\quad + \frac{a_4}{s}p(Tx_n, Sx_{n+1}) + \frac{a_5}{s}p(Tx_{n+1}, Sx_n) \\
 &= a_1p(y_n, y_{n+1}) + a_2p(y_n, y_{n+1}) + a_3p(y_{n+1}, y_{n+2}) \\
 &\quad + \frac{a_4}{s}p(y_{n+1}, y_{n+1}) + \frac{a_5}{s}p(y_{n+2}, y_n) \\
 &\preceq (a_1 + a_2)p(y_n, y_{n+1}) + a_3p(y_{n+1}, y_{n+2}) + \frac{a_4}{s}p(y_{n+1}, y_{n+1}) \\
 &\quad + a_5[p(y_n, y_{n+1}) + p(y_{n+1}, y_{n+2}) - p(y_{n+1}, y_{n+1})] \\
 &\preceq (a_1 + a_2 + a_5)p(y_n, y_{n+1}) + (a_3 + a_5)p(y_{n+1}, y_{n+2}) \\
 &\quad + \frac{(a_4 - a_5)}{s}p(y_{n+1}, y_{n+1}). \tag{3}
 \end{aligned}$$

Similarly,

$$\begin{aligned}
 p(y_{n+2}, y_{n+1}) &= p(Tx_{n+1}, Tx_n) \preceq \alpha(Sx_{n+1}, Sx_n)p(Tx_{n+1}, Tx_n) \\
 &\preceq a_1p(y_n, y_{n+1}) + a_2p(y_{n+1}, y_{n+2}) + a_3p(y_n, y_{n+1}) \\
 &\quad + \frac{a_4}{s}p(y_n, y_{n+2}) + \frac{a_5}{s}p(y_{n+1}, y_{n+1}) \\
 &\preceq (a_1 + a_3 + a_4)p(y_n, y_{n+1}) + (a_2 + a_4)p(y_{n+1}, y_{n+2}) \\
 &\quad + \frac{(a_5 - a_4)}{s}p(y_{n+1}, y_{n+1}). \tag{4}
 \end{aligned}$$

Hence, from (3) and (4) we have

$$p(y_{n+1}, y_{n+2}) \preceq \lambda p(y_n, y_{n+1}),$$

where $\lambda = (2a_1 + a_2 + a_3 + a_4 + a_5)/(2 - a_2 - a_3 - a_4 - a_5)$. Since $2a_1s + (s+1) \times (a_2 + a_3 + a_4 + a_5) < 2$, we have $0 \leq \lambda < 1/s$.

Similarly, we also have $p(y_n, y_{n+1}) \preceq \lambda p(y_{n-1}, y_n)$. Thus, for all $n \in \mathbb{N}$, by repetition of the above process n times we deduce that

$$p(y_n, y_{n+1}) \preceq \lambda p(y_{n-1}, y_n) \preceq \lambda^2 p(y_{n-2}, y_{n-1}) \preceq \cdots \preceq \lambda^n p(y_0, y_1).$$

Hence, for any $m, n \in \mathbb{Z}^+$ with $m > n$, it follows that

$$\begin{aligned}
 p(y_n, y_m) &\preceq s[p(y_n, y_{n+1}) + p(y_{n+1}, y_m)] \\
 &\preceq sp(y_n, y_{n+1}) + s^2[p(y_{n+1}, y_{n+2}) + p(y_{n+2}, y_m)] \preceq \cdots \\
 &\preceq sp(y_n, y_{n+1}) + s^2p(y_{n+1}, y_{n+2}) + \cdots \\
 &\quad + s^{m-n-1}[p(y_{m-2}, y_{m-1}) + p(y_{m-1}, y_m)] \\
 &\preceq sp(y_n, y_{n+1}) + s^2p(y_{n+1}, y_{n+2}) + \cdots \\
 &\quad + s^{m-n-1}p(y_{m-2}, y_{m-1}) + s^{m-n}p(y_{m-1}, y_m).
 \end{aligned}$$

Now, $p(y_n, y_{n+1}) \preceq \lambda^n p(y_0, y_1)$ for all $n \in \mathbb{N}$ and $0 \leq s\lambda < 1$ imply that

$$p(y_n, y_m) \preceq (s\lambda^n + s^2\lambda^{n+1} + \dots + s^{m-n}\lambda^{m-1})p(y_0, y_1) \preceq \frac{s\lambda^n}{1-s\lambda}p(y_0, y_1).$$

Let $\theta \ll c$ be given, choose $\delta > 0$ such that $c + N_\delta(\theta) \subseteq P$, where $N_\delta(\theta) = \{y \in E: \|y\| < \delta\}$. Also, choose a natural number N_1 such that $(\lambda^n/(1-\lambda))p(y_0, y_1) \in N_\delta(\theta)$ for all $n \geq N_1$. Then $(s\lambda^n/(-s\lambda))p(y_0, y_1) \ll c$ for all $n \geq N_1$. Thus,

$$p(y_n, y_m) \preceq \frac{s\lambda^n}{1-s\lambda}p(y_0, y_1) \ll c$$

for all $m > n \geq N_1$. Hence, $\{y_n\}$ is a θ -Cauchy sequence in (X, p) . Since (X, p) is a θ -complete partial cone b -metric space, there exists $y^* \in X$ such that $\{y_n\}$ converges to y^* with respect p and $p(y^*, y^*) = \theta$. Since $S(Y) = S(\bigcup_{i=1}^m A_i) = \bigcup_{i=1}^m S(A_i)$ is closed and $\{y_n\} \subset S(Y)$, we obtain that $y^* \in S(Y)$. Hence, there exists $z \in Y$ such that $y^* = Sz$.

Now, we will prove that $Tz = Sz$. For this, we have

$$\frac{1}{s}p(Tz, Sz) \preceq p(Tz, Tx_n) + p(Tx_n, Sz) = p(Tz, Tx_n) + p(y_{n+1}, y^*). \quad (5)$$

As $Y = \bigcup_{i=1}^m A_i$ is a cyclic representation of Y with respect to S and T , the sequence $\{x_n\}$ has infinite terms in each A_i for $i \in \{1, 2, \dots, m\}$. First, suppose that $z \in A_i$, then $y^* \in S(A_i)$, $Tz \in S(A_{i+1})$, and we take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with $x_{n_k} \in A_{i-1}$ (the existence of this subsequence is guaranteed by above mentioned argument).

Since $\alpha(y_n, y_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $y_n \rightarrow y^*$ as $n \rightarrow \infty$, by (iii) we have $\alpha(y_n, y^*) = \alpha(Sx_n, Sz) \geq 1$ for sufficiently large n . Hence, we also have $\alpha(Sz, Sx_n) \geq 1$ for sufficiently large n . For $x_{n_k} \in A_{i-1}$ and $z \in A_i$, by (1) we have

$$\begin{aligned} p(Tx_{n_k}, Tz) &\preceq \alpha(Sx_{n_k}, Sz)p(Tx_{n_k}, Tz) \\ &\preceq a_1p(y_{n_k}, y^*) + a_2p(y_{n_k}, y_{n_k+1}) + a_3p(Tz, Sz) \\ &\quad + \frac{a_4}{s}p(y_{n_k+1}, y^*) + \frac{a_5}{s}p(Tz, y_{n_k}) \\ &\preceq a_1p(y_{n_k}, y^*) + a_2p(y_{n_k}, y_{n_k+1}) + a_3p(Tz, Sz) \\ &\quad + \frac{a_4}{s}p(y_{n_k+1}, y^*) + a_5[p(Tz, Sz) + p(y^*, y_{n_k})] \end{aligned} \quad (6)$$

and, similarly,

$$\begin{aligned} p(Tz, Tx_{n_k}) &\preceq \alpha(Sz, Sx_{n_k})p(Tz, Tx_{n_k}) \\ &\preceq a_1p(y_{n_k}, y^*) + a_2p(Tz, Sz) + a_3p(y_{n_k}, y_{n_k+1}) \\ &\quad + \frac{a_4}{s}p(Tz, y_{n_k}) + \frac{a_5}{s}p(y_{n_k+1}, y^*) \\ &\preceq a_1p(y_{n_k}, y^*) + a_2p(Tz, Sz) + a_3p(y_{n_k}, y_{n_k+1}) \\ &\quad + a_4[p(Tz, Sz) + p(y^*, y_{n_k})] \\ &\quad + \frac{a_5}{s}p(y_{n_k+1}, y^*). \end{aligned} \quad (7)$$

From (5)–(7) we get

$$\begin{aligned} & \left(\frac{2}{s} - a_2 - a_3 - a_4 - a_5 \right) p(Tz, Sz) \\ & \preceq (2a_1 + a_4 + a_5)p(y_{n_k}, y^*) + (a_2 + a_3)p(y_{n_k}, y_{n_k+1}) \\ & \quad + \left(\frac{a_4 + a_5}{s} + 2 \right) p(y_{n_k+1}, y^*). \end{aligned} \quad (8)$$

Since $2a_1s + (s+1)(a_2 + a_3 + a_4 + a_5) < 2$, it follows that $q = 2/s - a_2 - a_3 - a_4 - a_5 > 0$. Let $M = \max\{2a_1 + a_4 + a_5, a_2 + a_3, (a_4 + a_5)/s + 2\}/q$. Since $y_n \rightarrow y^*$, for each $c \gg \theta$, choose a natural number N_2 such that $p(y_{n_k}, y_{n_k+1}) \ll c/(3M)$ and $p(y_{n_k}, y^*) \ll c/(3M)$ for all $k \geq N_2$. Hence, for $k \geq N_2$, it follows from (8) that

$$p(Tz, Sz) \ll \frac{c}{3} + \frac{c}{3} + \frac{c}{3} = c,$$

which implies that $p(Tz, Sz) = \theta$. Then $Tz = Sz$. Hence, S and T have a coincidence point in X .

Now, we will prove that S and T have a unique point of coincidence. Suppose that $Tw = Sw$ is another point of coincidence of S and T with $Tz = Sz \neq Tw = Sw$, then $\alpha(Sz, Sw) \geq 1$. By (1) we have

$$\begin{aligned} p(Tz, Tw) & \preceq \alpha(Sz, Sw)p(Tz, Tw) \\ & \preceq a_1p(Sz, Sw) + a_2p(Tz, Sz) + a_3p(Tw, Sw) \\ & \quad + \frac{a_4}{s}p(Tz, Sw) + \frac{a_5}{s}p(Tw, Sz) \\ & = a_1p(Tz, Tw) + a_2p(Tz, Tz) + a_3p(Tw, Tw) \\ & \quad + \frac{a_4}{s}p(Tz, Tw) + \frac{a_5}{s}p(Tw, Tz) \\ & \preceq \left(a_1 + \frac{a_4 + a_5}{s} \right) p(Tz, Tw) + a_2p(Tz, Tz) + a_3p(Tw, Tw) \\ & \preceq (a_1 + a_2 + a_3 + a_4 + a_5)p(Tz, Tw). \end{aligned}$$

Since $2a_1s + (s+1)(a_2 + a_3 + a_4 + a_5) < 2$, we get $\sum_{i=1}^5 a_i < 1$. Thus, $p(Tz, Tw) = \theta$. This is a contradiction. Hence, $Tz = Sz = Tw = Sw = v$ is a unique point of coincidence of S and T . Since S and T are weakly compatible, we have $Sv = STz = TSz = Tv$. Thus, $v = Sv = Tv$ by the uniqueness of point of coincidence of S and T . Hence, $Tz = Sz$ is the unique common fixed point of S and T . \square

Remark 2. Taking $\alpha(x, y) \equiv 1$, in Theorem 1, it extends and generalizes Theorem 2 in [4] and Theorem 12 in [2]. If we take $m = 1$ and $A_1 = X$ in Theorem 1, then we get immediately the following corollary.

Corollary 1 [α_S -Hardy-Rogers type]. Let (X, p) be a θ -complete partial cone b -metric space over a solid cone P of a normed vector space $(E, \|\cdot\|)$, $\alpha : X \times X \rightarrow [0, +\infty)$ be

a symmetric function. Suppose that the mappings $S, T : X \rightarrow X$ satisfy the contractive condition

$$\alpha(Sx, Sy)p(Tx, Ty) \preccurlyeq a_1p(Sx, Sy) + a_2p(Tx, Sx) + a_3p(Ty, Sy) + \frac{a_4}{s}p(Tx, Sy) + \frac{a_5}{s}p(Ty, Sx)$$

for all $x, y \in X$, where $a_i \geq 0$, $i = 1, 2, \dots, 5$, and $2a_1s + (s+1)(a_2+a_3+a_4+a_5) < 2$. Also assume that the following conditions hold:

- (i) $T(X) \subseteq S(X)$, $S(X)$ is closed, S and T are generalized α -admissible;
- (ii) there exists $x_0 \in X$ such that $\alpha(Sx_0, Tx_0) \geq 1$;
- (iii) if $\{y_n\}$ is a sequence in X such that $\alpha(y_n, y_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $y_n \rightarrow y$ as $n \rightarrow \infty$, then $\alpha(y_n, y) \geq 1$ for sufficiently large n .

Then S and T have a coincidence point in X , that is, there exists $z \in X$ such that $Sz = Tz$.

Remark 3. Taking $\alpha(x, y) \equiv 1$ in Corollary 1, it extends and generalizes Theorem 3.3 of [32] and so also the famous Hardy and Rogers fixed point theorem to that in the setting of partial cone b -metric spaces.

Example 6. Let $E = \mathbb{R}^2$, $P = \{(x, y) \in E : x, y \geq 0\}$, and $X = \mathbb{R}^+$. Define $p : X \times X \rightarrow P$ by

$$p(x, y) = \begin{cases} (|x - y|^2, |x - y|^2), & x, y \in [0, 4), \\ (\max\{x, y\}, |x - y|^2) & \text{otherwise} \end{cases}$$

for all $x, y \in X$. Then (X, p) is a θ -complete partial cone b -metric space with coefficient $s = 2$.

Suppose that $A_1 = [0, 3]$, $A_2 = [3, 4]$, $A_3 = [0, 3]$ and $Y = \bigcup_{i=1}^3 A_i = [0, 4]$. Define $S, T : Y \rightarrow Y$ and $\alpha : X \times X \rightarrow [0, +\infty)$ by

$$Sx = x, \quad Tx = \begin{cases} 3, & x \in [0, 4), \\ 1, & x = 4, \end{cases}$$

$$\alpha(x, y) = \begin{cases} 1/4, & x \in [0, 3], y = 4 \text{ or } x = 4, y \in [0, 3], \\ 1 & \text{otherwise.} \end{cases}$$

It is not difficult to prove that conditions (i)–(iii) of Theorem 1 are satisfied. Let $a_i = 1/9$ for $i = 1, 2, \dots, 5$. Now, we verify inequality (1) in Theorem 1. We consider the following four cases:

Case 1. If $x \in A_1$, $y \in A_2$, then, for $x \in [0, 3]$ and $y \in [3, 4)$, we have $|Tx - Ty|^2 = 1 - 1 = 0$, which implies that (1) holds.

Case 2. If $x \in A_1$, $y \in A_2$, then, for $x \in [0, 3]$ and $y = 4$, we have

$$\begin{aligned} \max\{Sx, Sy\} &= 4, & \max\{Sx, Tx\} &\geq 3, & \max\{Sy, Ty\} &= 4, \\ \max\{Sx, Ty\} &\geq 1, & \max\{Sy, Tx\} &= 4, \\ |Sx - Sy|^2 &\geq 1, & |Sx - Tx|^2 &\geq 0, & |Sy - Ty|^2 &= 9, \\ |Sx - Ty|^2 &\geq 0, & |Sy - Tx|^2 &= 1. \end{aligned}$$

Hence,

$$\begin{aligned} \alpha(Sx, Sy) \max\{Tx, Ty\} &= \frac{3}{4} \leq \frac{1}{9} \left(4 + 3 + 4 + \frac{1}{2} + 2 \right) \\ &\leq \frac{1}{9} \left(\max\{Sx, Sy\} + \max\{Sx, Tx\} + \max\{Sy, Ty\} \right. \\ &\quad \left. + \frac{\max\{Sx, Ty\}}{2} + \frac{\max\{Sy, Tx\}}{2} \right) \end{aligned}$$

and

$$\begin{aligned} \alpha(x, y) |Tx - Ty|^2 &= \frac{1}{4} |Tx - Ty|^2 = 1 \\ &\leq \frac{1}{9} \left(|Sx - Sy|^2 + |Sx - Tx|^2 + |Sy - Ty|^2 \right. \\ &\quad \left. + \frac{|Sx - Ty|^2}{2} + \frac{|Sy - Tx|^2}{2} \right), \end{aligned}$$

which implies that (1) holds.

Case 3. If $x \in A_2$, $y \in A_3 = A_1$. As in the previous case, we also have (1) holds.

Case 4. If $x \in A_3$, $y \in A_1$, then, for $x \in [0, 3]$ and $y \in [0, 3]$, we have $|Tx - Ty| = 0$, which implies that (1) holds.

Thus, all the conditions of Theorem 1 are satisfied. Therefore, S and T have a common fixed point in Y , indeed, $x = 3$ is a common fixed point of S and T .

Now, we introduce the concepts of mapping ψ and generalized cyclic $(\alpha, \psi)_S$ -Meir-Keeler contractive mappings.

Definition 9. Let (X, p) be a partial cone b -metric space with coefficient $s \geq 1$. Let Ψ stand for the family of nondecreasing mappings $\psi : P \rightarrow P$ such that

- (i) $\psi(\theta) = \theta$ and $\theta \prec \psi(x) \prec x/s$ for $x \in P \setminus \{\theta\}$;
- (ii) $\sum_{n=1}^{\infty} \|s^n \psi^n(x)\| < +\infty$ for each $x \in P$, where ψ^n is the n th iterate of ψ .

Example 7. Let $E = C[0, 1]$ with the norm $\|u\| = \max_{t \in [0, 1]} |u(t)|$ and $X = P = \{u \in E : u(t) \geq 0, t \in [0, 1]\}$. Define $\psi_1, \psi_2 : P \rightarrow P$, $p : X \times X \rightarrow P$ by

$$\psi_1(x) = \frac{x}{4}, \quad \psi_2(x) = \begin{cases} x/4, & 0 \leq \|x\| < 1; \\ 2x/7 & \text{otherwise,} \end{cases} \quad p(x, y) = \begin{cases} x^2, & x = y, \\ (x + y)^2 & \text{otherwise.} \end{cases}$$

It is clear that $\psi_1, \psi_2 \in \Psi$. Note that ψ_1, ψ_2 are examples of continuous and discontinuous functions in Ψ .

Definition 10. Let (X, p) be a partial cone b -metric space with coefficient $s \geq 1$, $\alpha : X \times X \rightarrow [0, +\infty)$ be a function, m be a positive integer, A_1, A_2, \dots, A_m be nonempty subsets of X , $Y = \bigcup_{i=1}^m A_i$ and $S, T : Y \rightarrow Y$ be two mappings. $T : Y \rightarrow Y$ is said to be a generalized cyclic $(\alpha, \psi)_S$ -Meir-Keeler contractive if Y is a cyclic representation of Y with respect to S and T for any x and y lying in different adjacently labeled sets A_i and A_{i+1} , $i = 1, 2, \dots, m$,

$$\alpha(Sx, Sy)p(Tx, Ty) \preceq \psi(u(x, y)), \quad (9)$$

where $A_{m+1} = A_1$, $\psi \in \Psi$ and $u(x, y) \in \{p(Sx, Sy), p(Tx, Sx), p(Ty, Sy), p(Tx, Sy)\}$.

Theorem 2. Let (X, p) be a θ -complete partial cone b -metric space over a solid cone P of a normed vector space $(E, \|\cdot\|)$, m be a positive integer, A_1, A_2, \dots, A_m be nonempty subsets of X , $Y = \bigcup_{i=1}^m A_i$, $T : X \rightarrow X$, be a generalized $(\alpha, \psi)_S$ -Meir-Keeler contractive mapping satisfying (9). Suppose that the following conditions hold:

- (i) S and T are generalized α -admissible;
- (ii) there exists $x_0 \in A_1$ such that $\alpha(Sx_0, Tx_0) \geq 1$;
- (iii) if $\{y_n\}$ is a sequence in X such that $\alpha(y_n, y_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $y_n \rightarrow y$ as $n \rightarrow \infty$, then $\alpha(y_n, y) \geq 1$, for sufficiently large n .

Then S and T have a coincidence point in X , that is, there exists $z \in X$ such that $Sz = Tz$. Moreover, if S, T are weakly compatible and Λ is the set of coincidence points of S and T , for all $x, y \in \Lambda$, we have $\alpha(Sx, Sy) \geq 1$. Then S and T have a unique common fixed point in X .

Proof. From the proof of Theorem 1 we can construct two sequences $\{x_n\}, \{y_n\} \subseteq X$ such that $y_{n+1} = Sx_{n+1} = Tx_n$ for all $n \in \mathbb{N}$, and there exists $i_n \in \{1, 2, \dots, m\}$ such that $x_n \in A_{i_n}$ and $x_{n+1} \in A_{i_{n+1}}$. By condition (ii) we deduce

$$\alpha(y_n, y_{n+1}) = \alpha(Sx_n, Sx_{n+1}) \geq 1 \quad \text{for all } n \in \mathbb{N}. \quad (10)$$

Since $x_n \in A_{i_n}$ and $x_{n+1} \in A_{i_{n+1}}$, by (9) and (10) we have

$$\begin{aligned} p(y_n, y_{n+1}) &= p(Tx_{n-1}, Tx_n) \preceq \alpha(Sx_{n-1}, Sx_n)p(Tx_{n-1}, Tx_n) \\ &\preceq \psi(u(x_{n-1}, x_n)), \end{aligned}$$

where

$$\begin{aligned} u(x_{n-1}, x_n) &\in \{p(Sx_{n-1}, Sx_n), p(Tx_{n-1}, Sx_{n-1}), p(Tx_n, Sx_n), p(Tx_{n-1}, Sx_n)\} \\ &= \{p(y_{n-1}, y_n), p(y_n, y_{n-1}), p(y_{n+1}, y_n), p(y_n, y_n)\} \\ &= \{p(y_{n-1}, y_n), p(y_{n+1}, y_n), p(y_n, y_n)\}. \end{aligned}$$

Thus, we get the following cases:

Case 1. $p(y_n, y_{n+1}) \preceq \psi(p(y_{n-1}, y_n))$.

Case 2. $p(y_n, y_{n+1}) \preceq \psi(p(y_n, y_{n+1}))$, it follows from Definition 9(i) that $p(y_n, y_{n+1}) = \theta$.

Case 3. $p(y_n, y_{n+1}) \preceq \psi(p(y_n, y_n)) \preceq \psi(p(y_{n-1}, y_n))$.

Then, in all cases, we have $p(y_n, y_{n+1}) \preceq \psi(p(y_{n-1}, y_n))$. Therefore,

$$p(y_n, y_{n+1}) \preceq \psi(p(y_{n-1}, y_n)) \preceq \psi^2(p(y_{n-2}, y_{n-1})) \preceq \cdots \preceq \psi^n(p(y_0, y_1)).$$

For $n, m \in \mathbb{Z}^+$ with $m > n$, it follows that

$$\begin{aligned} p(y_n, y_m) &\preceq sp(y_n, y_{n+1}) + s^2p(y_{n+1}, y_{n+2}) + \cdots + s^{m-n}p(y_{m-1}, y_m) \\ &\preceq s\psi^n(p(y_0, y_1)) + s^2\psi^{n+1}(p(y_0, y_1)) + \cdots + s^{m-n}\psi^{m-1}(p(y_0, y_1)) \\ &\preceq s^n\psi^n(p(y_0, y_1)) + s^{n+1}\psi^{n+1}(p(y_0, y_1)) + \cdots + s^{m-1}\psi^{m-1}(p(y_0, y_1)) \\ &= \sum_{k=n}^{m-1} s^k\psi^k(p(y_0, y_1)). \end{aligned}$$

Since $\sum_{n=1}^{\infty} \|s^n\psi^n(x)\| < +\infty$ for all $x \in P$, then for any given $\varepsilon > 0$, there exists $N_1 \in \mathbb{Z}^+$ such that $\sum_{k=n}^{\infty} \|s^k\psi^k(x)\| < \varepsilon$ for all $n > N_1$. Hence,

$$\left\| \sum_{k=n}^{m-1} s^k\psi^k(p(y_0, y_1)) \right\| \leq \sum_{k=n}^{m-1} \|s^k\psi^k(p(y_0, y_1))\| \leq \sum_{k=n}^{\infty} \|s^k\psi^k(p(y_0, y_1))\| < \varepsilon.$$

Thus, $\sum_{k=n}^{m-1} s^k\psi^k(p(y_0, y_1)) \rightarrow_{\|\cdot\|} \theta$. It follows from Lemma 1 that for any $c \gg \theta$, there exists $n_0 \in \mathbb{N}$ such that $\sum_{k=n}^{m-1} s^k\psi^k(p(x_0, x_1)) \ll c$ for all $n \geq n_0$. Thus,

$$p(y_n, y_m) \preceq \sum_{k=n}^{m-1} s^k\psi^k(p(y_0, y_1)) \ll c$$

for all $n, m \geq n_0$. Therefore, $\{y_n\}$ is a θ -Cauchy sequence. Since (X, p) is a θ -complete partial cone b -metric space, there exists $y^* \in X$ such that $y_n \rightarrow y^*$ and $p(y^*, y^*) = \theta$. Since $S(Y) = S(\bigcup_{i=1}^m A_i) = \bigcup_{i=1}^m S(A_i)$ is closed and $\{y_n\} \subset S(Y)$, we know that $y^* \in S(Y)$. Hence, there exists $z \in Y$ such that $y^* = Sz$.

Now, we prove that $Tz = Sz$. As $Y = \bigcup_{i=1}^m A_i$ is a cyclic representation of Y with respect to S and T , the sequence $\{x_n\}$ has infinite terms in each A_i for $i \in \{1, 2, \dots, m\}$. First, suppose that $z \in A_i$, then $y^* \in S(A_i)$, $Tz \in S(A_{i+1})$, and we take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with $x_{n_k} \in A_{i-1}$ (the existence of this subsequence is guaranteed by above mentioned argument).

Since $\alpha(y_n, y_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $y_n \rightarrow y^*$ as $n \rightarrow \infty$, by (iii) we have $\alpha(y_n, y^*) = \alpha(Sx_n, Sz) \geq 1$ for sufficiently large n . Let $c \gg \theta$, choose a natural

number N_2 such that $p(y_{n_k}, y_{n_k+1}) \ll c/2$ and $p(y_{n_k}, y^*) \ll c/2$ for all $k \geq N_2$. For $x_{n_k} \in A_{i-1}$ and $z \in A_i$, by (9) we have

$$\begin{aligned} \frac{1}{s}p(Tz, Sz) &\preceq p(Tx_{n_k}, Sz) + p(Tx_{n_k}, Tz) \\ &\preceq p(y_{n_k+1}, y^*) + \alpha(Sx_{n_k}, Sz)p(Tx_{n_k}, Tz) \\ &\preceq p(y_{n_k+1}, y^*) + \psi(u(x_{n_k}, z)), \end{aligned}$$

where

$$\begin{aligned} u(x_{n_k}, z) &\in \{p(Sx_{n_k}, Sz), p(Tx_{n_k}, Sx_{n_k}), p(Tz, Sz), p(Tx_{n_k}, Sz)\} \\ &= \{p(y_{n_k}, y^*), p(y_{n_k}, y_{n_k+1}), p(Tz, Sz), p(y_{n_k+1}, y^*)\}. \end{aligned}$$

Thus, we get the following cases:

Case 1. $(1/s)p(Tz, Sz) \preceq p(y_{n_k+1}, y^*) + \psi(p(y_{n_k}, y^*)) \preceq p(y_{n_k+1}, y^*) + (1/s) \times p(y_{n_k}, y^*) \ll c$ implies $p(Tz, Sz) = \theta$;

Case 2. $(1/s)p(Tz, Sz) \preceq p(y_{n_k+1}, y^*) + \psi(p(y_{n_k}, y_{n_k+1})) \preceq p(y_{n_k+1}, y^*) + (1/s)p(y_{n_k}, y_{n_k+1}) \ll c$ implies $p(Tz, Sz) = \theta$;

Case 3. $(1/s)p(Tz, Sz) \preceq p(y_{n_k+1}, y^*) + \psi(p(Tz, Sz)) \ll c + \psi(p(Tz, Sz))$ implies $(1/s)p(Tz, Sz) \preceq \psi(p(Tz, Sz))$. If $p(Tz, Sz) \neq \theta$, then $(1/s)p(Tz, Sz) \preceq \psi(p(Tz, Sz)) \prec (1/s)p(Tz, Sz)$. This is a contradiction. Thus, $p(Tz, Sz) = \theta$;

Case 4. $(1/s)p(Tz, Sz) \preceq p(y_{n_k+1}, y^*) + \psi(p(y_{n_k+1}, y^*)) \preceq p(y_{n_k+1}, y^*) + (1/s) \times p(y_{n_k+1}, y^*) \ll c$ implies $p(Tz, Sz) = \theta$.

Then, in all cases, we have $p(Tz, Sz) = \theta$. Hence, $Tz = Sz$. Thus, S and T have a coincidence point in X .

Now, we prove that S and T have a unique point of coincidence. Suppose that $Tw = Sw$ is another point of coincidence of S and T with $Tz = Sz \neq Tw = Sw$, then $\alpha(Sz, Sw) \geq 1$. By (9) we have

$$p(Tz, Tw) \preceq \alpha(Sz, Sw)p(Tz, Tw) \preceq \psi(u(z, w)),$$

where

$$\begin{aligned} u(x, y) &\in \{p(Sz, Sw), p(Tz, Sz), p(Tw, Sw), p(Tz, Sw)\} \\ &= \{p(Tz, Tw), p(Tz, Tz), p(Tw, Tw), p(Tz, Tw)\} \\ &= \{p(Tz, Tw), p(Tz, Tz), p(Tw, Tw)\}. \end{aligned}$$

Thus, we get the following cases:

Case 1. $p(Tz, Tw) \preceq \psi(p(Tz, Tw))$ implies $p(Tz, Tw) = \theta$.

Case 2. $p(Tz, Tw) \preceq \psi(p(Tz, Tz)) \preceq \psi(p(Tz, Tw))$ implies $p(Tz, Tw) = \theta$.

Case 3. $p(Tz, Tw) \preceq \psi(p(Tw, Tw)) \preceq \psi(p(Tz, Tw))$ implies $p(Tz, Tw) = \theta$.

Then, in all cases, we have $p(Tz, Tw) = \theta$. This is a contradiction. Hence, $Tz = Sz = Tw = Sw = v$ is a unique point of coincidence of S and T . Since S and T are weakly compatible, we have $Sv = STz = TSz = Tv$. Thus, $v = Sv = Tv$ by uniqueness of point of coincidence of S and T . Hence, $Tz = Sz$ is the unique common fixed point of S and T . \square

Remark 4. If we take $\alpha(x, y) \equiv 1$ in Theorem 2, it extends and generalizes Theorem 1 in [4], Theorem 9 in [2]. If we take $m = 1$ and $A_1 = X$ in Theorem 2, then it extends and generalizes Theorem 3.1 in [32], Theorems 15 and 17 in [18].

In Theorem 2, taking $\psi(x) = \lambda x$ with $0 \leq \lambda < 1/s$, we get the following corollary.

Corollary 2. Let (X, p) be a partial cone b -metric space with coefficient $s \geq 1$, $\alpha : X \times X \rightarrow [0, +\infty)$ be a function, m be a positive integer, A_1, A_2, \dots, A_m be nonempty subsets of X , $Y = \bigcup_{i=1}^m A_i$, and $S, T : Y \rightarrow Y$ be two mappings. Y is a cyclic representation of Y with respect to S and T for any x and y lying in different adjacently labeled sets A_i and A_{i+1} , $i = 1, 2, \dots, m$,

$$\alpha(Sx, Sy)p(Tx, Ty) \preceq \lambda u(x, y),$$

where $A_{m+1} = A_1$, $0 \leq \lambda < 1/s$, and $u(x, y) \in \{p(Sx, Sy), p(Tx, Sx), p(Ty, Sy), p(Tx, Ty)\}$. Also assume that the following conditions hold:

- (i) S and T are generalized α -admissible;
- (ii) there exists $x_0 \in A_1$ such that $\alpha(Sx_0, Tx_0) \geq 1$;
- (iii) if $\{y_n\}$ is a sequence in X such that $\alpha(y_n, y_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $y_n \rightarrow y$ as $n \rightarrow \infty$, then $\alpha(y_n, y) \geq 1$ for sufficiently large n .

Then S and T have a coincidence point in X , that is, there exists $z \in X$ such that $Sz = Tz$.

Remark 5. If we take $\alpha(x, y) \equiv 1$, in Corollary 2, it extends and generalizes Theorem 14 in [34].

Now, in order to support the usability of our results, we present the following example.

Example 8. Let $E = C[0, 1]$ with the norm $\|u\| = \max_{t \in [0, 1]} |u(t)|$ and $X = P = \{u \in E : u(t) \geq 0, t \in [0, 1]\}$. Define a mapping $p : X \times X \rightarrow P$ by

$$p(x, y) = \begin{cases} x^2, & x = y, \\ (x + y)^2 & \text{otherwise.} \end{cases}$$

Then (X, p) is a θ -complete partial cone b -metric space with coefficient $s = 3$. Define $S, T : X \rightarrow X$, $\alpha : X \times X \rightarrow [0, +\infty)$ by

$$Sx(t) = \frac{1}{2} \int_0^t x(s) ds, \quad Tx(t) = \frac{1}{6} \int_0^t x(s) ds, \quad \alpha(x, y) = \begin{cases} 1, & x \preceq y, \\ 0 & \text{otherwise.} \end{cases}$$

It is not difficult to prove that (i)–(iii) of Theorem 2 are satisfied. Define $\psi : P \rightarrow P$ by

$$\psi(x) = \begin{cases} x/4, & 0 \leq \|x\| < 1, \\ 2x/7 & \text{otherwise.} \end{cases}$$

Let $Ax(t) = \int_0^t x(s) \, ds$. Then $Sx = (1/2)Ax$ and $Tx = (1/6)Ax$. Let $m = 1$, $A_1 = X$. Now, we verify inequality (9) in Theorem 2. We consider the following three cases:

Case 1. If $x \gg y$, then $\alpha(x, y) = 0$ and $\alpha(Sx, Sy) = 0$, which implies that (9) holds.

Case 2. If $x = y = \theta$, then $p(Tx, Ty) = \theta$, that is, (9) holds.

Case 3. If $x = y \neq \theta$, then $\alpha(Sx, Sy) = 1$, and we have

$$\alpha(Sx, Sy)p(Tx, Ty) = \left(\frac{1}{6}Ax\right)^2 \preceq \frac{2}{27}(Ax)^2 = \frac{1}{6}p(Tx, Sx) \preceq \psi(p(Tx, Sx)),$$

which implies that (9) holds.

Case 4. If $x \neq y$ and $x \ll y$, then $\alpha(Sx, Sy) = 1$, and we have

$$\begin{aligned} \alpha(Sx, Sy)p(Tx, Ty) &= \left(\frac{1}{6}Ax + \frac{1}{6}Ay\right)^2 \preceq \frac{1}{24}(Ax + Ay)^2 \\ &= \frac{1}{6}p(Sx, Sy) \preceq \psi(p(Sx, Sy)), \end{aligned}$$

which implies that (9) holds.

Thus, all the conditions of Theorem 2 are satisfied. Therefore, S and T have a coincidence point in X , indeed, $x = \theta$ is a coincidence point of S and T .

4 An application: the existence of a common solution to integral equations

In this section, we apply Theorem 2 to study the existence of solutions to a class of system of nonlinear integral equations.

We consider the following system of integral equations

$$\begin{aligned} x(t) &= \int_0^t f(s, x(s)) \, ds, \\ x(t) &= \int_0^t x(s) \, ds \end{aligned} \tag{11}$$

for all $t \in I = [0, T]$, where $T > 0$, $f : I \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function.

Let $X = E = C(I, \mathbb{R})$ be the set of all real continuous functions on I and $P = \{u \in E : u \geq 0\}$. We endow X with the partial cone b -metric

$$p(x, y)(t) = e^t \max_{t \in [0, T]} |x(t) - y(t)|^2$$

for all $x, y \in X$. It is clear that (X, p) is a θ -complete partial cone b -metric space with coefficient $s = 2$. We endow X with the partial order \preceq given by

$$x \preceq y \quad \text{if and only if } x(t) \leq y(t) \text{ for all } t \in [0, T].$$

Let $u, v \in E$ such that $u \preceq v$,

$$\int_0^t u(s) \, ds \geq \int_0^t f(s, v(s)) \, ds \quad \text{and} \quad \int_0^t v(s) \, ds \geq \int_0^t f(s, u(s)) \, ds$$

for all $t \in [0, T]$. Let $A_1 = \{x \in E : x \preceq v\}$, $A_2 = \{x \in E : x \preceq u\}$ be two closed subsets of X and $Y = A_1 \cup A_2$.

Now, define the mappings $S, T : Y \rightarrow Y$ by

$$Sx(t) = \int_0^t x(s) \, ds, \quad Tx(t) = \int_0^t f(s, x(s)) \, ds$$

for all $x \in Y$. Then the existence of a solution to (11) is equivalent to the existence of a common fixed point of S and T .

Theorem 3. Suppose that the following hypotheses hold:

- (i) for all $s \in [0, T]$, $f(s, \cdot)$ is a nondecreasing function, that is, for all $t_1, t_2 \in \mathbb{R}$, $t_1 \leq t_2$ implies that $f(s, t_1) \leq f(s, t_2)$;
- (ii) if $f(s, x(s)) = x(s)$ for all $s \in [0, T]$, then we have

$$f\left(s, \int_0^s x(w) \, dw\right) = \int_0^s f(w, x(w)) \, dw \quad \text{for all } s \in [0, T];$$

- (iii) there exists a continuous function $K : [0, T] \rightarrow \mathbb{R}^+$ such that

$$f(s, y(s)) - f(s, x(s)) \leq K(s)[y(s) - x(s)]$$

for all $s \in [0, T]$, x and y lying in different adjacently labeled sets A_1 and A_2 with $x \preceq y$;

- (iv) there exists $L \in [0, 1)$ such that $\sup_{s \in [0, T]} K(s) \leq L/2$.

Then the integral equation (11) has a solution $x^* \in X$.

Proof. From condition (i) it is easy to verify that Y is a cyclic representation of Y with respect to S and T . By condition (ii) we can prove that S and T are weakly compatible. Define $\psi : P \rightarrow P$, $\alpha : X \times X \rightarrow [0, +\infty)$ by

$$\psi(x) = \frac{L^2}{3}x, \quad \alpha(x, y) = \begin{cases} 1, & x \preceq y, \\ 0 & \text{otherwise.} \end{cases}$$

Then S and T are generalized α -admissible, and (ii)–(iii) of Theorem 2 are satisfied. Now, for all x and y lying in different adjacently labeled sets A_1 and A_2 with $x \preceq y$, by (iii) and (iv) we have

$$\begin{aligned} & |Tx(t) - Ty(t)|^2 \\ &= \left| \int_0^t f(s, x(s)) \, ds - \int_0^t f(s, y(s)) \, ds \right|^2 = \left[\int_0^t (f(s, y(s)) - f(s, x(s))) \, ds \right]^2 \\ &\leq \left[\int_0^t K(s)(y(s) - x(s)) \, ds \right]^2 \leq \left[\sup_{s \in [0, t]} K(s) \int_0^t (y(s) - x(s)) \, ds \right]^2 \\ &\leq \left[\sup_{t \in [0, T]} K(t) \right]^2 \left[\int_0^t (y(s) - x(s)) \, ds \right]^2 \leq \frac{L^2}{4} |Sx(t) - Sy(t)|^2. \end{aligned}$$

Hence, for all $t \in [0, T]$, we have

$$\begin{aligned} & \alpha(Sx, Sy)p(Tx, Ty)(t) \\ &= \alpha(Sx, Sy)e^t \max_{t \in [0, T]} |Tx(t) - Ty(t)|^2 \leq \frac{L^2}{4} \left(e^t \max_{t \in [0, T]} |Sx(t) - Sy(t)|^2 \right) \\ &= \frac{L^2}{4} p(Sx, Sy)(t) \leq \lambda u(x, y)(t), \end{aligned}$$

where $\lambda = L^2/3 \in [0, 1/s)$ and $u(x, y) \in \{p(Sx, Sy), p(Tx, Sx), p(Ty, Sy), p(Tx, Sy)\}$. It follows that (9) holds.

Thus, all the conditions of Theorem 2 are satisfied. Then S and T have a common fixed point $x^* \in X$, that is, x^* is a solution of the system of integral equations (11). \square

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